

CHAPTER 24 THE THEORY OF MATRICES AND DETERMINANTS

EXERCISE 105 Page 279

1. Determine $\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} + \begin{pmatrix} 5 & 2 \\ -1 & 6 \end{pmatrix}$

$$\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} + \begin{pmatrix} 5 & 2 \\ -1 & 6 \end{pmatrix} = \begin{pmatrix} 3+5 & -1+2 \\ -4-1 & 7+6 \end{pmatrix} = \begin{pmatrix} 8 & 1 \\ -5 & 13 \end{pmatrix}$$

2. Determine $\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix} + \begin{pmatrix} 3 & 6 & 2 \\ 5 & -3 & 7 \\ -1 & 0 & 2 \end{pmatrix}$

$$\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix} + \begin{pmatrix} 3 & 6 & 2 \\ 5 & -3 & 7 \\ -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} (4+3) & (-7+6) & (6+2) \\ (-2+5) & (4-3) & (0+7) \\ (5-1) & (7+0) & (-4+2) \end{pmatrix} = \begin{pmatrix} 7 & -1 & 8 \\ 3 & 1 & 7 \\ 4 & 7 & -2 \end{pmatrix}$$

3. Determine $\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} - \begin{pmatrix} 5 & 2 \\ -1 & 6 \end{pmatrix}$

$$\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} - \begin{pmatrix} 5 & 2 \\ -1 & 6 \end{pmatrix} = \begin{pmatrix} 3-5 & -1-2 \\ -4-(-1) & 7-6 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ -3 & 1 \end{pmatrix}$$

4. Determine $\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} + \begin{pmatrix} 5 & 2 \\ -1 & 6 \end{pmatrix} - \begin{pmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{pmatrix}$

$$\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} + \begin{pmatrix} 5 & 2 \\ -1 & 6 \end{pmatrix} - \begin{pmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{pmatrix} = \begin{pmatrix} (3+5-(-1.3)) & (-1+2-7.4) \\ (-4+(-1)-2.5) & (7+6-(-3.9)) \end{pmatrix} \\ = \begin{pmatrix} 9.3 & -6.4 \\ -7.5 & 16.9 \end{pmatrix}$$

5. Determine $5\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} + 6\begin{pmatrix} 5 & 2 \\ -1 & 6 \end{pmatrix}$

$$5\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} + 6\begin{pmatrix} 5 & 2 \\ -1 & 6 \end{pmatrix} = \begin{pmatrix} 15 & -5 \\ -20 & 35 \end{pmatrix} + \begin{pmatrix} 30 & 12 \\ -6 & 36 \end{pmatrix} = \begin{pmatrix} 15+30 & -5+12 \\ -20+(-6) & 35+36 \end{pmatrix} = \begin{pmatrix} \mathbf{45} & \mathbf{7} \\ \mathbf{-26} & \mathbf{71} \end{pmatrix}$$

6. Determine $2\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix} + 3\begin{pmatrix} 3 & 6 & 2 \\ 5 & -3 & 7 \\ -1 & 0 & 2 \end{pmatrix} - 4\begin{pmatrix} 3.1 & 2.4 & 6.4 \\ -1.6 & 3.8 & -1.9 \\ 5.3 & 3.4 & -4.8 \end{pmatrix}$

$$2\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix} + 3\begin{pmatrix} 3 & 6 & 2 \\ 5 & -3 & 7 \\ -1 & 0 & 2 \end{pmatrix} - 4\begin{pmatrix} 3.1 & 2.4 & 6.4 \\ -1.6 & 3.8 & -1.9 \\ 5.3 & 3.4 & -4.8 \end{pmatrix} = \begin{pmatrix} (8+9-12.4) & (-14+18-9.6) & (12+6-25.6) \\ (-4+15+6.4) & (8-9-15.2) & (0+21+7.6) \\ (10-3-21.2) & (14+0-13.6) & (-8+6+19.2) \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{4.6} & \mathbf{-5.6} & \mathbf{-7.6} \\ \mathbf{17.4} & \mathbf{-16.2} & \mathbf{28.6} \\ \mathbf{-14.2} & \mathbf{0.4} & \mathbf{17.2} \end{pmatrix}$$

7. Determine $\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} \times \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

$$\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} \times \begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} (3 \times -2) + (-1 \times 5) \\ (-4 \times -2) + (7 \times 5) \end{pmatrix} = \begin{pmatrix} \mathbf{-11} \\ \mathbf{43} \end{pmatrix}$$

8. Determine $\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} \times \begin{pmatrix} 5 & 2 \\ -1 & 6 \end{pmatrix}$

$$\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} \times \begin{pmatrix} 5 & 2 \\ -1 & 6 \end{pmatrix} = \begin{pmatrix} (15+1) & (6-6) \\ (-20-7) & (-8+42) \end{pmatrix} = \begin{pmatrix} \mathbf{16} & \mathbf{0} \\ \mathbf{-27} & \mathbf{34} \end{pmatrix}$$

9. Determine $\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} \times \begin{pmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{pmatrix}$

$$\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} \times \begin{pmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{pmatrix} = \begin{pmatrix} (3 \times -1.3 + -1 \times 2.5) & (3 \times 7.4 + -1 \times -3.9) \\ (-4 \times -1.3 + 7 \times 2.5) & (-4 \times 7.4 + 7 \times -3.9) \end{pmatrix}$$

$$= \begin{pmatrix} -6.4 & 26.1 \\ 22.7 & -56.9 \end{pmatrix}$$

10. Determine $\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix} \times \begin{pmatrix} 4 \\ -11 \\ 7 \end{pmatrix}$

$$\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix} \times \begin{pmatrix} 4 \\ -11 \\ 7 \end{pmatrix} = \begin{pmatrix} (16+77+42) \\ (-8-44+0) \\ (20-77-28) \end{pmatrix} = \begin{pmatrix} 135 \\ -52 \\ -85 \end{pmatrix}$$

11. Determine $\begin{pmatrix} 3 & 6 & 2 \\ 5 & -3 & 7 \\ -1 & 0 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} 3 & 6 & 2 \\ 5 & -3 & 7 \\ -1 & 0 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} (3+0+2) & (0+6+0) \\ (5+0+7) & (0-3+0) \\ (-1+0+2) & (0+0+0) \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 12 & -3 \\ 1 & 0 \end{pmatrix}$$

12. Determine $\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix} \times \begin{pmatrix} 3.1 & 2.4 & 6.4 \\ -1.6 & 3.8 & -1.9 \\ 5.3 & 3.4 & -4.8 \end{pmatrix}$

$$\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix} \times \begin{pmatrix} 3.1 & 2.4 & 6.4 \\ -1.6 & 3.8 & -1.9 \\ 5.3 & 3.4 & -4.8 \end{pmatrix} = \begin{pmatrix} (12.4+11.2+31.8) & (9.6-26.6+20.4) & (25.6+13.3-28.8) \\ (-6.2-6.4+0) & (-4.8+15.2+0) & (-12.8-7.6+0) \\ (15.5-11.2-21.2) & (12+26.6-13.6) & (32-13.3+19.2) \end{pmatrix} = \begin{pmatrix} 55.4 & 3.4 & 10.1 \\ -12.6 & 10.4 & -20.4 \\ -16.9 & 25.0 & 37.9 \end{pmatrix}$$

$$\mathbf{13.} \text{ Show that } \begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} \times \begin{pmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{pmatrix} \neq \begin{pmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{pmatrix} \times \begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} \times \begin{pmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{pmatrix} = \begin{pmatrix} \mathbf{-6.4} & \mathbf{26.1} \\ \mathbf{22.7} & \mathbf{-56.9} \end{pmatrix} \text{ from Problem 9}$$

$$\begin{pmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{pmatrix} \times \begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} = \begin{pmatrix} (-1.3 \times 3 + 7.4 \times -4) & (-1.3 \times -1 + 7.4 \times 7) \\ (2.5 \times 3 + -3.9 \times -4) & (2.5 \times -1 + -3.9 \times 7) \end{pmatrix} = \begin{pmatrix} \mathbf{-33.5} & \mathbf{53.1} \\ \mathbf{23.1} & \mathbf{-29.8} \end{pmatrix}$$

$$\text{Hence, } \begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} \times \begin{pmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{pmatrix} \neq \begin{pmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{pmatrix} \times \begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix}$$

EXERCISE 106 Page 280

1. Calculate the determinant of $\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix}$

$$\begin{vmatrix} 3 & -1 \\ -4 & 7 \end{vmatrix} = (3)(7) - (-1)(-4) = 21 - 4 = \mathbf{17}$$

2. Calculate the determinant of $\begin{pmatrix} -2 & 5 \\ 3 & -6 \end{pmatrix}$

$$\begin{vmatrix} -2 & 5 \\ 3 & -6 \end{vmatrix} = (-2)(-6) - (5)(3) = 12 - 15 = \mathbf{-3}$$

3. Calculate the determinant of $\begin{pmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{pmatrix}$

$$\begin{vmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{vmatrix} = (-1.3)(-3.9) - (7.4)(2.5) = 5.07 - 18.5 = \mathbf{-13.43}$$

4. Evaluate $\begin{vmatrix} j2 & -j3 \\ (1+j) & j \end{vmatrix}$

$$\begin{vmatrix} j2 & -j3 \\ (1+j) & j \end{vmatrix} = (j2)(j) - (-j3)(1+j) = j^2 2 + j3(1+j) = -2 + j3 + j^2 3 = -2 + j3 - 3 = \mathbf{-5 + j3}$$

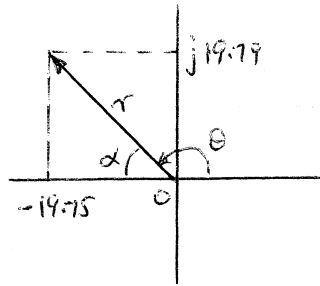
5. Evaluate $\begin{vmatrix} 2\angle 40^\circ & 5\angle -20^\circ \\ 7\angle -32^\circ & 4\angle -117^\circ \end{vmatrix}$

$$\begin{aligned} \begin{vmatrix} 2\angle 40^\circ & 5\angle -20^\circ \\ 7\angle -32^\circ & 4\angle -117^\circ \end{vmatrix} &= (2\angle 40^\circ)(4\angle -117^\circ) - (5\angle -20^\circ)(7\angle -32^\circ) \\ &= 8\angle -77^\circ - 35\angle -52^\circ = (1.800 - j7.795) - (21.548 - j27.580) \\ &= \mathbf{(-19.75 + j19.79)} \end{aligned}$$

From the diagram below, $r = \sqrt{(19.75^2 + 19.79^2)} = 27.96$ and $\alpha = \tan^{-1}\left(\frac{19.79}{19.75}\right) = 45.06^\circ$

and

$$\theta = 180^\circ - 45.06^\circ = 134.94^\circ$$



Hence, $\begin{vmatrix} 2\angle 40^\circ & 5\angle -20^\circ \\ 7\angle -32^\circ & 4\angle -117^\circ \end{vmatrix} = (-19.75 + j19.79) \text{ or } 27.96\angle 134.94^\circ$

6. Given matrix $A = \begin{pmatrix} (x-2) & 6 \\ 2 & (x-3) \end{pmatrix}$, determine values of x for which $|A| = 0$

If $|A| = 0$ then $\begin{vmatrix} (x-2) & 6 \\ 2 & (x-3) \end{vmatrix} = 0$

i.e. $(x-2)(x-3) - (6)(2) = 0$

i.e. $x^2 - 3x - 2x + 6 - 12 = 0$

i.e. $x^2 - 5x - 6 = 0$

i.e. $(x-6)(x+1) = 0$ by factorising

from which, $(x-6) = 0$ or $(x+1) = 0$

i.e. $x = 6$ or $x = -1$

EXERCISE 107 Page 281

1. Determine the inverse of $\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix}$

$$\begin{vmatrix} 3 & -1 \\ -4 & 7 \end{vmatrix} = (3)(7) - (-1)(-4) = 21 - 4 = 17$$

Hence, the inverse of $\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix}$ is: $\frac{1}{17} \begin{pmatrix} 7 & 1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} \frac{7}{17} & \frac{1}{17} \\ \frac{4}{17} & \frac{3}{17} \end{pmatrix}$

2. Determine the inverse of $\begin{pmatrix} \frac{1}{2} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{3}{5} \end{pmatrix}$

$$\begin{vmatrix} \frac{1}{2} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{3}{5} \end{vmatrix} = -\frac{3}{10} - \frac{2}{9} = \frac{2}{9} - \frac{3}{10} = \frac{20 - 27}{90} = -\frac{7}{90}$$

Hence, the inverse of $\begin{pmatrix} \frac{1}{2} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{3}{5} \end{pmatrix}$ is: $\frac{1}{-\frac{7}{90}} \begin{pmatrix} -\frac{3}{5} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{90}{7} \left(-\frac{3}{5}\right) & -\frac{90}{7} \left(-\frac{2}{3}\right) \\ -\frac{90}{7} \left(\frac{1}{3}\right) & -\frac{90}{7} \left(\frac{1}{2}\right) \end{pmatrix}$

$$= \begin{pmatrix} \frac{54}{7} & \frac{60}{7} \\ -\frac{30}{7} & -\frac{45}{7} \end{pmatrix} = \begin{pmatrix} 7\frac{5}{7} & 8\frac{4}{7} \\ -4\frac{2}{7} & -6\frac{3}{7} \end{pmatrix}$$

3. Determine the inverse of $\begin{pmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{pmatrix}$

The inverse of $\begin{pmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{pmatrix}$ is: $\frac{1}{(-1.3)(-3.9) - (7.4)(2.5)} \begin{pmatrix} -3.9 & -7.4 \\ -2.5 & -1.3 \end{pmatrix}$

$$\begin{aligned}
&= \frac{1}{-13.43} \begin{pmatrix} -3.9 & -7.4 \\ -2.5 & -1.3 \end{pmatrix} = \begin{pmatrix} \frac{3.9}{13.43} & \frac{7.4}{13.43} \\ \frac{2.5}{13.43} & \frac{1.3}{13.43} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{0.290} & \mathbf{0.551} \\ \mathbf{0.186} & \mathbf{0.097} \end{pmatrix}
\end{aligned}$$

EXERCISE 108 Page 282

1. Find the matrix of minors of $\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$

Matrix of minors of $\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix} = \begin{pmatrix} -16 & 8 & -34 \\ -14 & -46 & 63 \\ -24 & 12 & 2 \end{pmatrix}$

2. Find the matrix of cofactors of $\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$

Matrix of cofactors of $\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix} = \begin{pmatrix} -16 & -8 & -34 \\ 14 & -46 & -63 \\ -24 & -12 & 2 \end{pmatrix}$

3. Calculate the determinant of $\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$

$$\begin{vmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{vmatrix} = 4(-16) - (-7)(8) + 6(-34) \quad \text{using the top row}$$

$$= -64 + 56 - 204 = -212$$

4. Evaluate $\begin{vmatrix} 8 & -2 & -10 \\ 2 & -3 & -2 \\ 6 & 3 & 8 \end{vmatrix}$

$$\begin{vmatrix} 8 & -2 & -10 \\ 2 & -3 & -2 \\ 6 & 3 & 8 \end{vmatrix} = 8(-24 - -6) - 2(16 - -12) + -10(6 - -18) \quad \text{using the top row}$$

$$= -144 + 56 - 240 = -328$$

5. Calculate the determinant of $\begin{pmatrix} 3.1 & 2.4 & 6.4 \\ -1.6 & 3.8 & -1.9 \\ 5.3 & 3.4 & -4.8 \end{pmatrix}$

$$\begin{vmatrix} 3.1 & 2.4 & 6.4 \\ -1.6 & 3.8 & -1.9 \\ 5.3 & 3.4 & -4.8 \end{vmatrix} = 3.1(-18.24 + 6.46) - 2.4(7.68 + 10.07) + 6.4(-5.44 - 20.14) \quad \text{using the first row}$$

$$= 3.1(-11.78) - 2.4(17.75) + 6.4(-25.58) = -36.518 - 42.6 - 163.712 = -242.83$$

6. Evaluate $\begin{vmatrix} j2 & 2 & j \\ (1+j) & 1 & -3 \\ 5 & -j4 & 0 \end{vmatrix}$

$$\begin{vmatrix} j2 & 2 & j \\ (1+j) & 1 & -3 \\ 5 & -j4 & 0 \end{vmatrix} = (j2)(0 - j12) - 2(0 - -15) + j[(1+j)(-j4) - 5]$$

$$= j2(-j12) - 2(15) + j(-j4 + 4 - 5)$$

$$= 24 - 30 + 4 - j = -2 - j$$

7. Evaluate $\begin{vmatrix} 3\angle 60^\circ & j2 & 1 \\ 0 & (1+j) & 2\angle 30^\circ \\ 0 & 2 & j5 \end{vmatrix}$

$$\begin{vmatrix} 3\angle 60^\circ & j2 & 1 \\ 0 & (1+j) & 2\angle 30^\circ \\ 0 & 2 & j5 \end{vmatrix} = 3\angle 60^\circ [j5(1+j) - 4\angle 30^\circ] - j2(0 - 0) + 1(0 + 0) \quad \text{using the top row}$$

$$= 3\angle 60^\circ (j5 + j^2 5 - 4\angle 30^\circ) = 3\angle 60^\circ (j5 - 5 - (3.464 + j2))$$

$$= 3\angle 60^\circ (j5 - 5 - 3.464 - j2) = 3\angle 60^\circ (-8.464 + j3)$$

$$= 3\angle 60^\circ (8.98\angle 160.48^\circ) = 26.94\angle 220.48^\circ$$

$$= 26.94\angle -139.52^\circ \quad \text{or} \quad (-20.49 - j17.49)$$

8. Find the eigenvalues λ that satisfy the following equations:

$$(a) \begin{vmatrix} (2-\lambda) & 2 \\ -1 & (5-\lambda) \end{vmatrix} = 0 \quad (b) \begin{vmatrix} (5-\lambda) & 7 & -5 \\ 0 & (4-\lambda) & -1 \\ 2 & 8 & (-3-\lambda) \end{vmatrix} = 0$$

$$(a) \begin{vmatrix} (2-\lambda) & 2 \\ -1 & (5-\lambda) \end{vmatrix} = 0 \quad \text{hence, } (2-\lambda)(5-\lambda) - (-2) = 0$$

$$\text{i.e.} \quad 10 - 7\lambda + \lambda^2 + 2 = 0$$

$$\text{i.e.} \quad \lambda^2 - 7\lambda + 12 = 0$$

$$\text{and} \quad (\lambda - 4)(\lambda - 3) = 0$$

$$\text{from which,} \quad \lambda - 4 = 0 \quad \text{or} \quad \lambda - 3 = 0$$

Thus, **eigenvalues, $\lambda = 3$ or 4**

$$(b) \begin{vmatrix} (5-\lambda) & 7 & -5 \\ 0 & (4-\lambda) & -1 \\ 2 & 8 & (-3-\lambda) \end{vmatrix} = 0$$

$$\text{hence, } (5-\lambda)[(4-\lambda)(-3-\lambda)+8] - 7(0+2) - 5[0-2(4-\lambda)] = 0$$

$$\text{i.e.} \quad (5-\lambda)[-12-\lambda+\lambda^2+8] - 14 + 40 - 10\lambda = 0$$

$$\text{and} \quad (5-\lambda)(\lambda^2 - \lambda - 4) + 26 - 10\lambda = 0$$

$$\text{i.e.} \quad 5\lambda^2 - 5\lambda - 20 - \lambda^3 + \lambda^2 + 4\lambda + 26 - 10\lambda = 0$$

$$\text{and} \quad -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$$\text{or} \quad \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\text{Let } f(\lambda) = \lambda^3 - 6\lambda^2 + 11\lambda - 6$$

$$f(0) = -6$$

$$f(1) = 1 - 6 + 11 - 6 = 0 \quad \text{hence, } (\lambda - 1) \text{ is a factor}$$

$$f(2) = 8 - 24 + 22 - 6 = 0 \quad \text{hence, } (\lambda - 2) \text{ is a factor}$$

$$f(3) = 27 - 54 + 33 - 6 = 0 \quad \text{hence, } (\lambda - 3) \text{ is a factor}$$

Thus, $(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$

from which, **eigenvalues, $\lambda = 1$ or 2 or 3**

EXERCISE 109 Page 283

1. Write down the transpose of $\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$

The transpose of $\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$ is $\begin{pmatrix} 4 & -2 & 5 \\ -7 & 4 & 7 \\ 6 & 0 & -4 \end{pmatrix}$

2. Write down the transpose of $\begin{pmatrix} 3 & 6 & \frac{1}{2} \\ 5 & -\frac{2}{3} & 7 \\ -1 & 0 & \frac{3}{5} \end{pmatrix}$

The transpose of $\begin{pmatrix} 3 & 6 & \frac{1}{2} \\ 5 & -\frac{2}{3} & 7 \\ -1 & 0 & \frac{3}{5} \end{pmatrix}$ is $\begin{pmatrix} 3 & 5 & -1 \\ 6 & -\frac{2}{3} & 0 \\ \frac{1}{2} & 7 & \frac{3}{5} \end{pmatrix}$

3. Determine the adjoint of $\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$

Matrix of cofactors of $\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$ is: $\begin{pmatrix} -16 & -8 & -34 \\ 14 & -46 & -63 \\ -24 & -12 & 2 \end{pmatrix}$

Adjoint = transpose of cofactors = $\begin{pmatrix} -16 & 14 & -24 \\ -8 & -46 & -12 \\ -34 & -63 & 2 \end{pmatrix}$

4. Determine the adjoint of $\begin{pmatrix} 3 & 6 & \frac{1}{2} \\ 5 & -\frac{2}{3} & 7 \\ -1 & 0 & \frac{3}{5} \end{pmatrix}$

Matrix of cofactors of $\begin{pmatrix} 3 & 6 & \frac{1}{2} \\ 5 & -\frac{2}{3} & 7 \\ -1 & 0 & \frac{3}{5} \end{pmatrix}$ is: $\begin{pmatrix} -\frac{2}{5} & -10 & \frac{2}{3} \\ -3\frac{3}{5} & 2\frac{3}{10} & -6 \\ 42\frac{1}{3} & -18\frac{1}{2} & -32 \end{pmatrix}$

Adjoint = transpose of cofactors = $\begin{pmatrix} -\frac{2}{5} & -3\frac{3}{5} & 42\frac{1}{3} \\ -10 & 2\frac{3}{10} & -18\frac{1}{2} \\ \frac{2}{3} & -6 & -32 \end{pmatrix}$

5. Find the inverse of $\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$

From question 3 above, adjoint = $\begin{pmatrix} -16 & 14 & -24 \\ -8 & -46 & -63 \\ -34 & -63 & 2 \end{pmatrix}$

$$\begin{vmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{vmatrix} = 4(-16) + 7(8) + 6(-34) = -212$$

Hence, the inverse of $\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$ is: $-\frac{1}{212} \begin{pmatrix} -16 & 14 & -24 \\ -8 & -46 & -63 \\ -34 & -63 & 2 \end{pmatrix}$

6. Find the inverse of $\begin{pmatrix} 3 & 6 & \frac{1}{2} \\ 5 & -\frac{2}{3} & 7 \\ -1 & 0 & \frac{3}{5} \end{pmatrix}$

Matrix of cofactors of $\begin{pmatrix} 3 & 6 & \frac{1}{2} \\ 5 & -\frac{2}{3} & 7 \\ -1 & 0 & \frac{3}{5} \end{pmatrix}$ is: $\begin{pmatrix} -\frac{2}{5} & -10 & -\frac{2}{3} \\ -3\frac{3}{5} & 2\frac{3}{10} & -6 \\ 42\frac{1}{3} & -18\frac{1}{2} & -32 \end{pmatrix}$

Transpose of cofactors = adjoint = $\begin{pmatrix} -\frac{2}{5} & -3\frac{3}{5} & 42\frac{1}{3} \\ -10 & 2\frac{3}{10} & -18\frac{1}{2} \\ -\frac{2}{3} & -6 & -32 \end{pmatrix}$

$$\begin{vmatrix} 3 & 6 & \frac{1}{2} \\ 5 & -\frac{2}{3} & 7 \\ -1 & 0 & \frac{3}{5} \end{vmatrix} = 3\left(-\frac{2}{5}\right) - 6(3+7) + \frac{1}{2}\left(-\frac{2}{3}\right) = -\frac{6}{5} - \frac{60}{1} - \frac{1}{3} = \frac{-18-900-5}{15} = -\frac{923}{15}$$

Hence, the inverse of $\begin{pmatrix} 3 & 6 & \frac{1}{2} \\ 5 & -\frac{2}{3} & 7 \\ -1 & 0 & \frac{3}{5} \end{pmatrix}$ is: $\frac{1}{-\frac{923}{15}} \begin{pmatrix} -\frac{2}{5} & -3\frac{3}{5} & 42\frac{1}{3} \\ -10 & 2\frac{3}{10} & -18\frac{1}{2} \\ -\frac{2}{3} & -6 & -32 \end{pmatrix}$

$$= -\frac{15}{923} \begin{pmatrix} -\frac{2}{5} & -3\frac{3}{5} & 42\frac{1}{3} \\ -10 & 2\frac{3}{10} & -18\frac{1}{2} \\ -\frac{2}{3} & -6 & -32 \end{pmatrix}$$

CHAPTER 25 APPLICATIONS OF MATRICES AND DETERMINANTS

EXERCISE 110 Page 287

1. Use matrices to solve:

$$\begin{aligned} 3x + 4y &= 0 \\ 2x + 5y + 7 &= 0 \end{aligned}$$

$$3x + 4y = 0$$

$$2x + 5y = -7$$

Hence,

$$\begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -7 \end{pmatrix}$$

The inverse of $\begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$ is: $\frac{1}{15-8} \begin{pmatrix} 5 & -4 \\ -2 & 3 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 5 & -4 \\ -2 & 3 \end{pmatrix}$

Thus,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 5 & -4 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ -7 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 28 \\ -21 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

i.e. $x = 4$ and $y = -3$

2. Use matrices to solve:

$$\begin{aligned} 2p + 5q + 14.6 &= 0 \\ 3.1p + 1.7q + 2.06 &= 0 \end{aligned}$$

$$2p + 5q = -14.6$$

$$3.1p + 1.7q = -2.06$$

Hence,

$$\begin{pmatrix} 2 & 5 \\ 3.1 & 1.7 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -14.6 \\ -2.06 \end{pmatrix}$$

The inverse of $\begin{pmatrix} 2 & 5 \\ 3.1 & 1.7 \end{pmatrix}$ is: $\frac{1}{3.4-15.5} \begin{pmatrix} 1.7 & -5 \\ -3.1 & 2 \end{pmatrix} = \frac{1}{-12.1} \begin{pmatrix} 1.7 & -5 \\ -3.1 & 2 \end{pmatrix}$

Thus,

$$\begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{-12.1} \begin{pmatrix} 1.7 & -5 \\ -3.1 & 2 \end{pmatrix} \begin{pmatrix} -14.6 \\ -2.06 \end{pmatrix} = \frac{1}{-12.1} \begin{pmatrix} -14.52 \\ 41.14 \end{pmatrix} = \begin{pmatrix} 1.2 \\ -3.4 \end{pmatrix}$$

i.e. $p = 1.2$ and $q = -3.4$

3. Use matrices to solve: $x + 2y + 3z = 5$

$$2x - 3y - z = 3$$

$$-3x + 4y + 5z = 3$$

Since

$$x + 2y + 3z = 5$$

$$2x - 3y - z = 3$$

$$-3x + 4y + 5z = 3$$

then,

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -3 & -1 \\ -3 & 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix}$$

Matrix of cofactors is: $\begin{pmatrix} -11 & -7 & -1 \\ 2 & 14 & -10 \\ 7 & 7 & -7 \end{pmatrix}$ and the transpose of cofactors is: $\begin{pmatrix} -11 & 2 & 7 \\ -7 & 14 & 7 \\ -1 & -10 & -7 \end{pmatrix}$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & -3 & -1 \\ -3 & 4 & 5 \end{vmatrix} = 1(-11) - 2(7) + 3(-1) = -28$$

The inverse of $\begin{pmatrix} 1 & 2 & 3 \\ 2 & -3 & -1 \\ -3 & 4 & 5 \end{pmatrix}$ is: $\frac{1}{-28} \begin{pmatrix} -11 & 2 & 7 \\ -7 & 14 & 7 \\ -1 & -10 & -7 \end{pmatrix}$

$$\text{Thus, } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{28} \begin{pmatrix} -11 & 2 & 7 \\ -7 & 14 & 7 \\ -1 & -10 & -7 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix} = -\frac{1}{28} \begin{pmatrix} -28 \\ 28 \\ -56 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

i.e.

$$\mathbf{x = 1, y = -1 \text{ and } z = 2}$$

4. Use matrices to solve: $3a + 4b - 3c = 2$

$$-2a + 2b + 2c = 15$$

$$7a - 5b + 4c = 26$$

Since

$$3a + 4b - 3c = 2$$

$$-2a + 2b + 2c = 15$$

$$7a - 5b + 4c = 26$$

then,

$$\begin{pmatrix} 3 & 4 & -3 \\ -2 & 2 & 2 \\ 7 & -5 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 15 \\ 26 \end{pmatrix}$$

Matrix of cofactors is: $\begin{pmatrix} 18 & 22 & -4 \\ -1 & 33 & 43 \\ 14 & 0 & 14 \end{pmatrix}$ and the transpose of cofactors is: $\begin{pmatrix} 18 & -1 & 14 \\ 22 & 33 & 0 \\ -4 & 43 & 14 \end{pmatrix}$

$$\begin{vmatrix} 3 & 4 & -3 \\ -2 & 2 & 2 \\ 7 & -5 & 4 \end{vmatrix} = 3(18) - 4(-22) + -3(-4) = 154$$

The inverse of $\begin{pmatrix} 3 & 4 & -3 \\ -2 & 2 & 2 \\ 7 & -5 & 4 \end{pmatrix}$ is: $\frac{1}{154} \begin{pmatrix} 18 & -1 & 14 \\ 22 & 33 & 0 \\ -4 & 43 & 14 \end{pmatrix}$

Thus, $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = -\frac{1}{154} \begin{pmatrix} 18 & -1 & 14 \\ 22 & 33 & 0 \\ -4 & 43 & 14 \end{pmatrix} \begin{pmatrix} 2 \\ 15 \\ 26 \end{pmatrix} = \frac{1}{154} \begin{pmatrix} 385 \\ 539 \\ 1001 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 3.5 \\ 6.5 \end{pmatrix}$

i.e. **a = 2.5, b = 3.5 and c = 6.5**

5. Use matrices to solve: $p + 2q + 3r + 7.8 = 0$
 $2p + 5q - r - 1.4 = 0$
 $5p - q + 7r - 3.5 = 0$

Since $p + 2q + 3r = -7.8$
 $2p + 5q - r = 1.4$
 $5p - q + 7r = 3.5$

from which, $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & -1 \\ 5 & -1 & 7 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} -7.8 \\ 1.4 \\ 3.5 \end{pmatrix}$

Matrix of cofactors is: $\begin{pmatrix} 34 & -19 & -27 \\ -17 & -8 & 11 \\ -17 & 7 & 1 \end{pmatrix}$ and the transpose of cofactors is: $\begin{pmatrix} 34 & -17 & -17 \\ -19 & -8 & 7 \\ -27 & 11 & 1 \end{pmatrix}$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & -1 \\ 5 & -1 & 7 \end{vmatrix} = 1(34) - 2(19) + 3(-27) = -85$$

The inverse of $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & -1 \\ 5 & -1 & 7 \end{pmatrix}$ is: $\frac{1}{-85} \begin{pmatrix} 34 & -17 & -17 \\ -19 & -8 & 7 \\ -27 & 11 & 1 \end{pmatrix}$

$$\text{Thus, } \begin{pmatrix} p \\ q \\ r \end{pmatrix} = -\frac{1}{85} \begin{pmatrix} 34 & -17 & -17 \\ -19 & -8 & 7 \\ -27 & 11 & 1 \end{pmatrix} \begin{pmatrix} -7.8 \\ 1.4 \\ 3.5 \end{pmatrix} = -\frac{1}{85} \begin{pmatrix} -348.5 \\ 161.5 \\ 229.5 \end{pmatrix} = \begin{pmatrix} 4.1 \\ -1.9 \\ -2.7 \end{pmatrix}$$

$$\text{i.e. } \mathbf{p} = 4.1, \mathbf{q} = -1.9 \text{ and } \mathbf{r} = -2.7$$

6. In two closed loops of an electrical circuit, the currents flowing are given by the simultaneous equations

$$I_1 + 2I_2 + 4 = 0$$

$$5I_1 + 3I_2 - 1 = 0$$

Use matrices to solve for I_1 and I_2

$$I_1 + 2I_2 = -4$$

$$5I_1 + 3I_2 = 1$$

$$\text{Hence, } \begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

$$\text{The inverse of } \begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix} \text{ is: } \frac{1}{3-10} \begin{pmatrix} 3 & -2 \\ -5 & 1 \end{pmatrix} = -\frac{1}{7} \begin{pmatrix} 3 & -2 \\ -5 & 1 \end{pmatrix}$$

$$\text{Thus, } \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = -\frac{1}{7} \begin{pmatrix} 3 & -2 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ 1 \end{pmatrix} = -\frac{1}{7} \begin{pmatrix} -14 \\ 21 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\text{i.e. } \mathbf{I_1 = 2 \text{ and } I_2 = -3}$$

7. The relationship between the displacement, s , velocity, v , and acceleration, a , of a piston is given by the equations

$$s + 2v + 2a = 4$$

$$3s - v + 4a = 25$$

$$3s + 2v - a = -4$$

Use matrices to determine the values of s , v and a

$$\text{Since } s + 2v + 2a = 4$$

$$3s - v + 4a = 25$$

$$3s + 2v - a = -4$$

from which,
$$\begin{pmatrix} 1 & 2 & 2 \\ 3 & -1 & 4 \\ 3 & 2 & -1 \end{pmatrix} \begin{pmatrix} s \\ v \\ a \end{pmatrix} = \begin{pmatrix} 4 \\ 25 \\ -4 \end{pmatrix}$$

Matrix of cofactors is: $\begin{pmatrix} -7 & 15 & 9 \\ 6 & -7 & 4 \\ 10 & 2 & -7 \end{pmatrix}$ and the transpose of cofactors is: $\begin{pmatrix} -7 & 6 & 10 \\ 15 & -7 & 2 \\ 9 & 4 & -7 \end{pmatrix}$

$$\begin{vmatrix} 1 & 2 & 2 \\ 3 & -1 & 4 \\ 3 & 2 & -1 \end{vmatrix} = 1(-7) - 2(-15) + 2(9) = 41$$

The inverse of $\begin{pmatrix} 1 & 2 & 2 \\ 3 & -1 & 4 \\ 3 & 2 & -1 \end{pmatrix}$ is: $\frac{1}{41} \begin{pmatrix} -7 & 6 & 10 \\ 15 & -7 & 2 \\ 9 & 4 & -7 \end{pmatrix}$

Thus,
$$\begin{pmatrix} s \\ v \\ a \end{pmatrix} = \frac{1}{41} \begin{pmatrix} -7 & 6 & 10 \\ 15 & -7 & 2 \\ 9 & 4 & -7 \end{pmatrix} \begin{pmatrix} 4 \\ 25 \\ -4 \end{pmatrix} = \frac{1}{41} \begin{pmatrix} 82 \\ -123 \\ 164 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

i.e. $s = 2, v = -3$ and $a = 4$

8. In a mechanical system, acceleration $\ddot{x}$, velocity $\dot{x}$ and distance x are related by the

simultaneous equations:

$$\begin{aligned} 3.4\ddot{x} + 7.0\dot{x} - 13.2x &= -11.39 \\ -6.0\ddot{x} + 4.0\dot{x} + 3.5x &= 4.98 \\ 2.7\ddot{x} + 6.0\dot{x} + 7.1x &= 15.91 \end{aligned}$$

Use matrices to find the values of $\ddot{x}$, $\dot{x}$ and x .

Since

$$\begin{aligned} 3.4\ddot{x} + 7.0\dot{x} - 13.2x &= -11.39 \\ -6.0\ddot{x} + 4.0\dot{x} + 3.5x &= 4.98 \\ 2.7\ddot{x} + 6.0\dot{x} + 7.1x &= 15.91 \end{aligned}$$

then,
$$\begin{pmatrix} 3.4 & 7.0 & -13.2 \\ -6.0 & 4.0 & 3.5 \\ 2.7 & 6.0 & 7.1 \end{pmatrix} \begin{pmatrix} \ddot{x} \\ \dot{x} \\ x \end{pmatrix} = \begin{pmatrix} -11.39 \\ 4.98 \\ 15.91 \end{pmatrix}$$

Matrix of cofactors is:
$$\begin{pmatrix} 7.4 & 52.05 & -46.8 \\ -128.9 & 59.78 & -1.5 \\ 77.3 & 67.3 & 55.6 \end{pmatrix}$$

and the transpose of cofactors is:
$$\begin{pmatrix} 7.4 & -128.9 & 77.3 \\ 52.05 & 59.78 & 67.3 \\ -46.8 & -1.5 & 55.6 \end{pmatrix}$$

$$\begin{vmatrix} 3.4 & 7.0 & -13.2 \\ -6.0 & 4.0 & 3.5 \\ 2.7 & 6.0 & 7.1 \end{vmatrix} = 3.4(7.4) - 7.0(-52.05) + (-13.2)(-46.8) = 1007.27$$

The inverse of $\begin{pmatrix} 3.4 & 7.0 & -13.2 \\ -6.0 & 4.0 & 3.5 \\ 2.7 & 6.0 & 7.1 \end{pmatrix}$ is: $\frac{1}{1007.27} \begin{pmatrix} 7.4 & -128.9 & 77.3 \\ 52.05 & 59.78 & 67.3 \\ -46.8 & -1.5 & 55.6 \end{pmatrix}$

Thus,
$$\begin{pmatrix} \ddot{x} \\ \dot{x} \\ x \end{pmatrix} = \frac{1}{1007.27} \begin{pmatrix} 7.4 & -128.9 & 77.3 \\ 52.05 & 59.78 & 67.3 \\ -46.8 & -1.5 & 55.6 \end{pmatrix} \begin{pmatrix} -11.39 \\ 4.98 \\ 15.91 \end{pmatrix} = \frac{1}{1007.27} \begin{pmatrix} 503.635 \\ 775.5979 \\ 1410.178 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.77 \\ 1.4 \end{pmatrix}$$

i.e. $\ddot{x} = 0.5, \dot{x} = 0.77 \text{ and } x = 1.4$

EXERCISE 111 Page 291

- 1.** Use determinants to solve the simultaneous equations: $3x - 5y = -17.6$
 $7y - 2x - 22 = 0$

Since $3x - 5y + 17.6 = 0$
 $-2x + 7y - 22 = 0$

then
$$\frac{x}{\begin{vmatrix} -5 & 17.6 \\ 7 & -22 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & 17.6 \\ -2 & -22 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 3 & -5 \\ -2 & 7 \end{vmatrix}}$$

i.e.
$$\frac{x}{-13.2} = \frac{-y}{-30.8} = \frac{1}{11}$$

from which, $x = \frac{-13.2}{11} = -1.2$ and $y = \frac{30.8}{11} = 2.8$

- 2.** Use determinants to solve the simultaneous equations: $2.3m - 4.4n = 6.84$
 $8.5n - 6.7m = 1.23$

Since $2.3m - 4.4n - 6.84 = 0$
 $-6.7m + 8.5n - 1.23 = 0$

then
$$\frac{m}{\begin{vmatrix} -4.4 & -6.84 \\ 8.5 & -1.23 \end{vmatrix}} = \frac{-n}{\begin{vmatrix} 2.3 & -6.84 \\ -6.7 & -1.23 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 2.3 & -4.4 \\ -6.7 & 8.5 \end{vmatrix}}$$

i.e.
$$\frac{m}{63.552} = \frac{-n}{-48.657} = \frac{1}{-9.93}$$

from which, $m = \frac{63.552}{-9.93} = -6.4$ and $n = \frac{48.657}{-9.93} = -4.9$

- 3.** Use determinants to solve the simultaneous equations: $3x + 4y + z = 10$
 $2x - 3y + 5z + 9 = 0$
 $x + 2y - z = 6$

Since $3x + 4y + z - 10 = 0$
 $2x - 3y + 5z + 9 = 0$
 $x + 2y - z - 6 = 0$

$$\text{then} \quad \begin{array}{c} x \\ \left| \begin{array}{ccc} 4 & 1 & -10 \\ -3 & 5 & 9 \\ 2 & -1 & -6 \end{array} \right| \end{array} = \begin{array}{c} -y \\ \left| \begin{array}{ccc} 3 & 1 & -10 \\ 2 & 5 & 9 \\ 1 & -1 & -6 \end{array} \right| \end{array} = \begin{array}{c} z \\ \left| \begin{array}{ccc} 3 & 4 & -10 \\ 2 & -3 & 9 \\ 1 & 2 & -6 \end{array} \right| \end{array} = \begin{array}{c} -1 \\ \left| \begin{array}{ccc} 3 & 4 & 1 \\ 2 & -3 & 5 \\ 1 & 2 & -1 \end{array} \right| \end{array}$$

$$\text{i.e.} \quad \frac{x}{4(-21) - 1(0) - 10(-7)} = \frac{-y}{3(-21) - 1(-21) - 10(-7)} = \frac{z}{3(0) - 4(-21) - 10(7)} = \frac{-1}{3(-7) - 4(-7) + 1(7)}$$

$$\text{i.e.} \quad \frac{x}{-14} = \frac{-y}{28} = \frac{z}{14} = \frac{-1}{14}$$

$$\text{Hence,} \quad x = \frac{14}{14} = 1, \quad y = \frac{28}{14} = 2 \quad \text{and} \quad z = \frac{-14}{14} = -1$$

4. Use determinants to solve the simultaneous equations: $1.2p - 2.3q - 3.1r + 10.1 = 0$
 $4.7p + 3.8q - 5.3r - 21.5 = 0$
 $3.7p - 8.3q + 7.4r + 28.1 = 0$

$$\begin{aligned} \text{Since} \quad & 1.2p - 2.3q - 3.1r + 10.1 = 0 \\ & 4.7p + 3.8q - 5.3r - 21.5 = 0 \\ & 3.7p - 8.3q + 7.4r + 28.1 = 0 \end{aligned}$$

$$\text{then} \quad \begin{array}{c} p \\ \left| \begin{array}{ccc} -2.3 & -3.1 & 10.1 \\ 3.8 & -5.3 & -21.5 \\ -8.3 & 7.4 & 28.1 \end{array} \right| \end{array} = \begin{array}{c} -q \\ \left| \begin{array}{ccc} 1.2 & -3.1 & 10.1 \\ 4.7 & -5.3 & -21.5 \\ 3.7 & 7.4 & 28.1 \end{array} \right| \end{array} = \begin{array}{c} r \\ \left| \begin{array}{ccc} 1.2 & -2.3 & 10.1 \\ 4.7 & 3.8 & -21.5 \\ 3.7 & -8.3 & 28.1 \end{array} \right| \end{array} = \begin{array}{c} -1 \\ \left| \begin{array}{ccc} 1.2 & -2.3 & -3.1 \\ 4.7 & 3.8 & -5.3 \\ 3.7 & -8.3 & 7.4 \end{array} \right| \end{array}$$

$$\begin{aligned} & \frac{p}{-2.3(10.17) + 3.1(-71.67) + 10.1(-15.87)} = \frac{-q}{1.2(10.17) + 3.1(211.62) + 10.1(54.39)} \\ & = \frac{r}{1.2(-71.67) + 2.3(211.62) + 10.1(-53.07)} = \frac{-1}{1.2(-15.87) + 2.3(54.39) - 3.1(-53.07)} \end{aligned}$$

$$\text{i.e.} \quad \frac{p}{-405.855} = \frac{-q}{1217.565} = \frac{r}{-135.285} = \frac{-1}{270.57}$$

$$\text{Hence,} \quad p = \frac{405.855}{270.57} = 1.5, \quad q = \frac{1217.565}{270.57} = 4.5 \quad \text{and} \quad r = \frac{135.285}{270.57} = 0.5$$

5. Use determinants to solve the simultaneous equations:

$$\frac{x}{2} - \frac{y}{3} + \frac{2z}{5} = -\frac{1}{20}$$

$$\frac{x}{4} + \frac{2y}{3} - \frac{z}{2} = \frac{19}{40}$$

$$x + y - z = \frac{59}{60}$$

Since

$$\frac{x}{2} - \frac{y}{3} + \frac{2z}{5} + \frac{1}{20} = 0 \quad (1)$$

$$\frac{x}{4} + \frac{2y}{3} - \frac{z}{2} - \frac{19}{40} = 0 \quad (2)$$

$$x + y - z - \frac{59}{60} = 0 \quad (3)$$

Multiplying each term in (1) by 60 gives: $30x - 20y + 24z + 3 = 0$

Multiplying each term in (2) by 120 gives: $30x + 80y - 60z - 57 = 0$

Multiplying each term in (3) by 60 gives: $60x + 60y - 60z - 59 = 0$

then

$$\begin{vmatrix} x & -y & z & -1 \\ -20 & 24 & 3 & 1 \\ 80 & -60 & -57 & -19 \\ 60 & -60 & -59 & -59 \end{vmatrix} = \begin{vmatrix} 30 & 24 & 3 & 3 \\ 30 & -60 & -57 & -57 \\ 60 & -60 & -59 & -59 \end{vmatrix} = \begin{vmatrix} 30 & -20 & 3 & 3 \\ 30 & 80 & -57 & -57 \\ 60 & 60 & -59 & -59 \end{vmatrix} = \begin{vmatrix} 30 & -20 & 24 & 3 \\ 30 & 80 & -60 & -57 \\ 60 & 60 & -60 & -59 \end{vmatrix}$$

$$\frac{x}{-20(120) - 24(-1300) + 3(-1200)} = \frac{-y}{30(120) - 24(1650) + 3(1800)}$$

$$= \frac{z}{30(-1300) + 20(1650) + 3(-3000)} = \frac{-1}{30(-1200) + 20(1800) + 24(-3000)}$$

i.e.

$$\frac{x}{25200} = \frac{-y}{-30600} = \frac{z}{-15000} = \frac{-1}{-72000}$$

Hence,

$$x = \frac{25200}{72000} = \frac{7}{20}, \quad y = \frac{30600}{72000} = \frac{17}{40} \quad \text{and} \quad z = \frac{-15000}{72000} = -\frac{5}{24}$$

6. In a system of forces, the relationship between two forces F_1 and F_2 is given by

$$5F_1 + 3F_2 + 6 = 0$$

$$3F_1 + 5F_2 + 18 = 0$$

Use determinants to solve for F_1 and F_2

Since $5F_1 + 3F_2 + 6 = 0$

$$3F_1 + 5F_2 + 18 = 0$$

then
$$\frac{F_1}{\begin{vmatrix} 3 & 6 \\ 5 & 18 \end{vmatrix}} = \frac{-F_2}{\begin{vmatrix} 5 & 6 \\ 3 & 18 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 5 & 3 \\ 3 & 5 \end{vmatrix}}$$

i.e.
$$\frac{F_1}{24} = \frac{-F_2}{72} = \frac{1}{16}$$

from which,
$$F_1 = \frac{24}{16} = 1.5 \quad \text{and} \quad F_2 = -\frac{72}{16} = -4.5$$

7. Applying mesh-current analysis to an a.c. circuit results in the following equations:

$$(5 - j4)I_1 - (-j4)I_2 = 100\angle 0^\circ$$

$$(4 + j3 - j4)I_2 - (-j4)I_1 = 0$$

Solve the equations for I_1 and I_2 .

$$(5 - j4)I_1 - (-j4)I_2 - 100\angle 0^\circ = 0$$

$$-(-j4)I_1 + (4 + j3 - j4)I_2 + 0 = 0$$

i.e.
$$(5 - j4)I_1 + j4I_2 - 100 = 0$$

$$j4I_1 + (4 - j)I_2 + 0 = 0$$

Hence,
$$\frac{I_1}{\begin{vmatrix} j4 & -100 \\ (4 - j) & 0 \end{vmatrix}} = \frac{-I_2}{\begin{vmatrix} (5 - j4) & -100 \\ j4 & 0 \end{vmatrix}} = \frac{1}{\begin{vmatrix} (5 - j4) & j4 \\ j4 & (4 - j) \end{vmatrix}}$$

i.e.
$$\frac{I_1}{100(4 - j)} = \frac{-I_2}{j400} = \frac{1}{(5 - j4)(4 - j) - (j4)^2}$$

i.e.
$$\frac{I_1}{400 - j100} = \frac{I_2}{-j400} = \frac{1}{32 - j21}$$

Thus,
$$I_1 = \frac{400 - j100}{32 - j21} = \frac{412.31\angle -14.04^\circ}{38.275\angle -33.27^\circ} = 10.77\angle 19.23^\circ \text{ A}$$

and
$$I_2 = \frac{-j400}{32 - j21} = \frac{400\angle -90^\circ}{38.275\angle -33.27^\circ} = 10.45\angle -56.73^\circ \text{ A}$$

8. Kirchhoff's laws are used to determine the current equations in an electrical network and show

that $i_1 + 8i_2 + 3i_3 = -31$

$$3i_1 - 2i_2 + i_3 = -5$$

$$2i_1 - 3i_2 + 2i_3 = 6$$

Use determinants to solve for i_1 , i_2 and i_3

Since $i_1 + 8i_2 + 3i_3 + 31 = 0$

$$3i_1 - 2i_2 + i_3 + 5 = 0$$

$$2i_1 - 3i_2 + 2i_3 - 6 = 0$$

then

$$\begin{matrix} i_1 \\ 8 & 3 & 31 \end{matrix} = \begin{matrix} -i_2 \\ 1 & 3 & 31 \end{matrix} = \begin{matrix} i_3 \\ 1 & 8 & 31 \end{matrix} = \begin{matrix} -1 \\ 1 & 8 & 3 \end{matrix}$$

$$\begin{matrix} -2 & 1 & 5 \\ -3 & 2 & -6 \end{matrix} \quad \begin{matrix} 3 & 1 & 5 \\ 2 & 2 & -6 \end{matrix} \quad \begin{matrix} 3 & -2 & 5 \\ 2 & -3 & -6 \end{matrix} \quad \begin{matrix} 3 & -2 & 1 \\ 2 & -3 & 2 \end{matrix}$$

i.e.

$$\frac{i_1}{8(-16) - 3(27) + 31(-1)} = \frac{-i_2}{1(-16) - 3(-28) + 31(4)} = \frac{i_3}{1(27) - 8(-28) + 31(-5)} = \frac{-1}{1(-1) - 8(4) + 3(-5)}$$

i.e.

$$\frac{i_1}{-240} = \frac{-i_2}{192} = \frac{i_3}{96} = \frac{-1}{-48}$$

Hence,

$$i_1 = \frac{-240}{48} = -5, \quad i_2 = \frac{-192}{48} = -4 \quad \text{and} \quad i_3 = \frac{96}{48} = 2$$

9. The forces in three members of a framework are F_1 , F_2 and F_3 . They are related by the simultaneous equations shown below.

$$1.4 F_1 + 2.8 F_2 + 2.8 F_3 = 5.6$$

$$4.2 F_1 - 1.4 F_2 + 5.6 F_3 = 35.0$$

$$4.2 F_1 + 2.8 F_2 - 1.4 F_3 = -5.6$$

Find the values of F_1 , F_2 and F_3 using determinants.

$$1.4 F_1 + 2.8 F_2 + 2.8 F_3 - 5.6 = 0$$

$$4.2 F_1 - 1.4 F_2 + 5.6 F_3 - 35.0 = 0$$

$$4.2 F_1 + 2.8 F_2 - 1.4 F_3 + 5.6 = 0$$

$$\text{Hence, } \begin{array}{c} F_1 \\ \left[\begin{array}{ccc} 2.8 & 2.8 & -5.6 \\ -1.4 & 5.6 & -35.0 \\ 2.8 & -1.4 & 5.6 \end{array} \right] \end{array} = \begin{array}{c} -F_2 \\ \left[\begin{array}{ccc} 1.4 & 2.8 & -5.6 \\ 4.2 & 5.6 & -35.0 \\ 4.2 & -1.4 & 5.6 \end{array} \right] \end{array} = \begin{array}{c} F_3 \\ \left[\begin{array}{ccc} 1.4 & 2.8 & -5.6 \\ 4.2 & -1.4 & -35.0 \\ 4.2 & 2.8 & 5.6 \end{array} \right] \end{array} = \begin{array}{c} -1 \\ \left[\begin{array}{ccc} 1.4 & 2.8 & 2.8 \\ 4.2 & -1.4 & 5.6 \\ 4.2 & 2.8 & -1.4 \end{array} \right] \end{array}$$

$$\text{i.e. } \frac{F_1}{2.8(-17.64) - 2.8(90.16) - 5.6(-13.72)} = \frac{-F_2}{1.4(-17.64) - 2.8(170.52) - 5.6(-29.4)}$$

$$= \frac{F_3}{1.4(90.16) - 2.8(170.52) - 5.6(17.64)} = \frac{-1}{1.4(-13.72) - 2.8(-29.4) + 2.8(17.64)}$$

$$\text{i.e. } \frac{F_1}{-225.008} = \frac{-F_2}{-337.512} = \frac{F_3}{-450.016} = \frac{-1}{112.504}$$

$$\text{Thus, } F_1 = \frac{225.008}{112.504} = 2 \quad F_2 = \frac{-337.512}{112.504} = -3 \quad \text{and} \quad F_3 = \frac{450.016}{112.504} = 4$$

10. Mesh-current analysis produces the following three equations:

$$20\angle 0^\circ = (5 + 3 - j4)I_1 - (3 - j4)I_2$$

$$10\angle 90^\circ = (3 - j4 + 2)I_2 - (3 - j4)I_1 - 2I_3$$

$$-15\angle 0^\circ - 10\angle 90^\circ = (12 + 2)I_3 - 2I_2$$

Solve the equations for the loop currents I_1 , I_2 and I_3

$$\text{Rearranging gives: } (8 - j4)I_1 - (3 - j4)I_2 + 0I_3 - 20 = 0$$

$$-(3 - j4)I_1 + (5 - j4)I_2 - 2I_3 - j10 = 0$$

$$0I_1 - 2I_2 + 14I_3 + (15 + j10) = 0$$

Hence,

$$\begin{array}{c} I_1 \\ \left[\begin{array}{ccc} -(3-j4) & 0 & -20 \\ (5-j4) & -2 & -j10 \\ -2 & 14 & (15+j10) \end{array} \right] \end{array} = \begin{array}{c} -I_2 \\ \left[\begin{array}{ccc} (8-j4) & 0 & -20 \\ -(3-j4) & -2 & -j10 \\ 0 & 14 & (15+j10) \end{array} \right] \end{array} = \begin{array}{c} I_3 \\ \left[\begin{array}{ccc} (8-j4) & -(3-j4) & -20 \\ -(3-j4) & (5-j4) & -j10 \\ 0 & -2 & (15+j10) \end{array} \right] \end{array}$$

$$= \begin{array}{c} -1 \\ \left[\begin{array}{ccc} (8-j4) & -(3-j4) & 0 \\ -(3-j4) & (5-j4) & -2 \\ 0 & -2 & 14 \end{array} \right] \end{array}$$

i.e.

$$\frac{I_1}{-(3-j4)[-2(15+j10)+j140]-20[14(5-j4)-4]} = \frac{-I_2}{(8-j4)[-2(15+j10)+j140]-20[-14(3-j4)]}$$

$$= \frac{I_3}{(8-j4)[(5-j4)(15+j10)-j20]+(3-j4)[-(3-j4)(15+j10)]-20[2(3-j4)]}$$

$$= \frac{-1}{(8-j4)[14(5-j4)-4] + (3-j4)[-14(3-j4)]}$$

$$\begin{aligned} \text{i.e. } \frac{I_1}{-(3-j4)[-30+j120] - 20[66-j56]} &= \frac{-I_2}{(8-j4)[-30+j120] - 20[-42+j56]} \\ &= \frac{I_3}{(8-j4)[115-j30] + (3-j4)[-85+j30] - 40(3-j4)} \\ &= \frac{-1}{(8-j4)[66-j56] + (3-j4)[-42+j56]} \end{aligned}$$

$$\begin{aligned} \text{i.e. } \frac{I_1}{-(-90+j360+j120+480) - 1320+j1120} &= \frac{-I_2}{-240+j960+j120+480+840-j1120} \\ &= \frac{I_3}{920-j240-j460-120-255+j430+120-120+j160} \\ &= \frac{-1}{528-j448-j264-224-126+j168+j168+224} \end{aligned}$$

$$\text{i.e. } \frac{I_1}{(-1710+j640)} = \frac{-I_2}{(1080-j40)} = \frac{I_3}{(545-j110)} = \frac{-1}{(402-j376)}$$

$$\text{Hence, } I_1 = \frac{-(-1710+j640)}{(402-j376)} = \frac{1825.84\angle -20.52^\circ}{550.44\angle -43.09^\circ} = \mathbf{3.317\angle 22.57^\circ \text{ A}}$$

$$I_2 = \frac{(1080-j40)}{(402-j376)} = \frac{1080.74\angle -2.12^\circ}{550.44\angle -43.09^\circ} = \mathbf{1.963\angle 40.97^\circ \text{ A}}$$

$$I_3 = \frac{-(545-j110)}{(402-j376)} = \frac{-555.99\angle -11.41^\circ}{550.44\angle -43.09^\circ} = \frac{555.99\angle -191.41^\circ}{550.44\angle -43.09^\circ} = \mathbf{1.010\angle -148.32^\circ \text{ A}}$$

EXERCISE 112 Page 292

1. Repeat problems 3, 4, 5, 7 and 8 of Exercise 110 on page 287, using Cramers rule.

Just a sample of these questions are detailed below. The remainder are left to the reader.

2. Repeat problems 3, 4, 8 and 9 of Exercise 111 on page 291, using Cramers rule

Just a sample of these questions are detailed below. The remainder are left to the reader.

1. Q(3), Exercise 110

Use Cramers rule to solve: $x + 2y + 3z = 5$

$$2x - 3y - z = 3$$

$$-3x + 4y + 5z = 3$$

$$x = \frac{\begin{vmatrix} 5 & 2 & 3 \\ 3 & -3 & -1 \\ 3 & 4 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & -3 & -1 \\ -3 & 4 & 5 \end{vmatrix}} = \frac{5(-11) - 2(18) + 3(21)}{1(-11) - 2(7) + 3(-1)} = \frac{-28}{-28} = 1$$

$$y = \frac{\begin{vmatrix} 1 & 5 & 3 \\ 2 & 3 & -1 \\ -3 & 3 & 5 \end{vmatrix}}{-28} = \frac{1(18) - 5(7) + 3(15)}{-28} = \frac{28}{-28} = -1$$

$$z = \frac{\begin{vmatrix} 1 & 2 & 5 \\ 2 & -3 & 3 \\ -3 & 4 & 3 \end{vmatrix}}{-28} = \frac{1(-21) - 2(15) + 5(-1)}{-28} = \frac{-56}{-28} = 2$$

1. Q(7), Exercise 110

Use Cramers rule to solve: $s + 2v + 2a = 4$

$$3s - v + 4a = 25$$

$$3s + 2v - a = -4$$

$$\mathbf{s} = \frac{\begin{vmatrix} 4 & 2 & 2 \\ 25 & -1 & 4 \\ -4 & 2 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 2 \\ 3 & -1 & 4 \\ 3 & 2 & -1 \end{vmatrix}} = \frac{4(-7) - 2(-9) + 2(46)}{1(-7) - 2(-15) + 2(9)} = \frac{82}{41} = 2$$

$$\mathbf{v} = \frac{\begin{vmatrix} 1 & 4 & 2 \\ 3 & 25 & 4 \\ 3 & -4 & -1 \end{vmatrix}}{41} = \frac{1(-9) - 4(-15) + 2(-87)}{41} = \frac{-123}{41} = -3$$

$$\mathbf{a} = \frac{\begin{vmatrix} 1 & 2 & 4 \\ 3 & -1 & 25 \\ 3 & 2 & -4 \end{vmatrix}}{41} = \frac{1(-46) - 2(-87) + 4(9)}{41} = \frac{164}{41} = 4$$

2. Q(8), Exercise 111

Use Cramers rule to solve:

$$\begin{aligned} i_1 + 8i_2 + 3i_3 &= -31 \\ 3i_1 - 2i_2 + i_3 &= -5 \\ 2i_1 - 3i_2 + 2i_3 &= 6 \end{aligned}$$

$$i_1 = \frac{\begin{vmatrix} -31 & 8 & 3 \\ -5 & -2 & 1 \\ 6 & -3 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 8 & 3 \\ 3 & -2 & 1 \\ 2 & -3 & 2 \end{vmatrix}} = \frac{-31(-1) - 8(-16) + 3(27)}{1(-1) - 8(4) + 3(-5)} = \frac{240}{-48} = -5$$

$$i_2 = \frac{\begin{vmatrix} 1 & -31 & 3 \\ 3 & -5 & 1 \\ 2 & 6 & 2 \end{vmatrix}}{-48} = \frac{1(-16) + 31(4) + 3(28)}{-48} = \frac{192}{-48} = -4$$

$$i_3 = \frac{\begin{vmatrix} 1 & 8 & -31 \\ 3 & -2 & -5 \\ 2 & -3 & 6 \end{vmatrix}}{-48} = \frac{1(-27) - 8(28) - 31(-5)}{-48} = \frac{-96}{-48} = 2$$

1. In a mass-spring-damper system, the acceleration $\ddot{x}$ m/s², velocity $\dot{x}$ m/s and displacement x m are related by the following simultaneous equations:

$$6.2 \ddot{x} + 7.9 \dot{x} + 12.6x = 18.0$$

$$7.5 \ddot{x} + 4.8 \dot{x} + 4.8x = 6.39$$

$$13.0 \ddot{x} + 3.5 \dot{x} - 13.0x = -17.4$$

By using Gaussian elimination, determine the acceleration, velocity and displacement for the system, correct to 2 decimal places.

$$6.2 \ddot{x} + 7.9 \dot{x} + 12.6x = 18.0 \quad (1)$$

$$7.5 \ddot{x} + 4.8 \dot{x} + 4.8x = 6.39 \quad (2)$$

$$13.0 \ddot{x} + 3.5 \dot{x} - 13.0x = -17.4 \quad (3)$$

$$(2) - \frac{7.5}{6.2} \times (1) \text{ gives: } 0 - 4.7565 \dot{x} - 10.442x = -15.384 \quad (2')$$

$$(3) - \frac{13.0}{6.2} \times (1) \text{ gives: } 0 - 13.065 \dot{x} - 39.419x = -55.142 \quad (3')$$

$$(3') - \frac{-13.065}{-4.7565} \times (2') \text{ gives: } 0 + 0 - 10.737x = -12.886$$

from which,
$$x = \frac{-12.886}{-10.737} = 1.20$$

From (3'),
$$-13.065 \dot{x} - 39.419(1.2) = -55.142$$

i.e.
$$-13.065 \dot{x} = -55.142 + 39.419(1.2)$$

and
$$\dot{x} = \frac{-55.142 + 39.419(1.2)}{-13.065} = 0.60$$

From (1),
$$6.2 \ddot{x} + 7.9(0.60) + 12.6(1.2) = 18.0$$

$$6.2 \ddot{x} = 18.0 - 4.74 - 15.2 = -1.86$$

and
$$\ddot{x} = \frac{-1.86}{6.2} = -0.30$$

2. The tensions, T_1 , T_2 and T_3 in a simple framework are given by the equations:

$$5T_1 + 5T_2 + 5T_3 = 7.0$$

$$T_1 + 2T_2 + 4T_3 = 2.4$$

$$4T_1 + 2T_2 = 4.0$$

Determine T_1 , T_2 and T_3 using Gaussian elimination.

$$5T_1 + 5T_2 + 5T_3 = 7.0 \quad (1)$$

$$T_1 + 2T_2 + 4T_3 = 2.4 \quad (2)$$

$$4T_1 + 2T_2 + 0T_3 = 4.0 \quad (3)$$

$$(2) - \frac{1}{5} \times (1) \text{ gives: } 0 + T_2 + 3T_3 = 1.0 \quad (2')$$

$$(3) - \frac{4}{5} \times (1) \text{ gives: } 0 - 2T_2 - 4T_3 = -1.6 \quad (3')$$

$$(3') - \left(\frac{-2}{1} \right) \times (2') \text{ gives: } 2T_3 = 0.4$$

from which, $T_3 = 0.2$

$$\text{In } (3') \quad -2T_2 - 4(0.2) = -1.6$$

$$-2T_2 = -1.6 + 0.8 = -0.8$$

and $T_2 = 0.4$

$$\text{In } (1) \quad 5T_1 + 5(0.4) + 5(0.2) = 7.0$$

$$\text{i.e. } 5T_1 = 7.0 - 2.0 - 1.0 = 4.0$$

and $T_1 = 0.8$

3. Repeat problems 3, 4, 5, 7 and 8 of Exercise 110 on page 287, using the Gaussian elimination method.

These problems are left to the reader using the same method as in the above examples.

4. Repeat problems 3, 4, 8 and 9 of Exercise 111 on page 291, using the Gaussian elimination method.

These problems are left to the reader using the same method as in the above examples.

1. Determine the (a) eigenvalues (b) eigenvectors for the matrix: $\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$

(a) The eigenvalue is determined by solving the characteristic equation $|\mathbf{A} - \lambda \mathbf{I}| = 0$

$$\text{i.e.} \quad \left| \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0 \quad \text{i.e.} \quad \left| \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\text{i.e.} \quad \begin{vmatrix} 2-\lambda & -4 \\ -1 & -1-\lambda \end{vmatrix} = 0$$

(Given a square matrix, we can get used to going straight to this characteristic equation)

$$\text{Hence,} \quad (2 - \lambda)(-1 - \lambda) - (-4)(-1) = 0$$

$$\text{i.e.} \quad -2 - 2\lambda + \lambda + \lambda^2 - 4 = 0$$

$$\text{and} \quad \lambda^2 - \lambda - 6 = 0$$

$$\text{i.e.} \quad (\lambda - 3)(\lambda + 2) = 0$$

$$\text{from which,} \quad \lambda - 3 = 0 \quad \text{i.e.} \quad \lambda = 3 \quad \text{or} \quad \lambda + 2 = 0 \quad \text{i.e.} \quad \lambda = -2$$

(Instead of factorising, the quadratic formula could be used; even electronic calculators can solve quadratic equations).

Hence, the eigenvalues of the matrix $\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$ are 3 and -2

(b) From (a) the eigenvalues of $\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$ are $\lambda_1 = 3$ and $\lambda_2 = -2$

Using the equation $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda_1 = 3$

$$\text{then} \quad \begin{pmatrix} 2-3 & -4 \\ -1 & -1-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

from which,

$$-x_1 - 4x_2 = 0$$

from which,

$$x_1 = -4x_2$$

Hence, whatever value x_2 is, the value of x_1 will be - 4 times greater. Hence the simplest

eigenvector is: $\mathbf{x}_1 = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

Using the equation $(\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda_2 = -2$

then
$$\begin{pmatrix} 2-(-2) & -4 \\ -1 & -1-(-2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 4 & -4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

from which,

$$4x_1 - 4x_2 = 0$$

and

$$-x_1 + x_2 = 0$$

From either of these two equations, $x_1 = x_2$ or $x_2 = x_1$

Hence, whatever value x_1 is, the value of x_2 will be the same. Hence the simplest eigenvector

is: $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Summarising, $\mathbf{x}_1 = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_1 = 3$

and $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_2 = -2$

2. Determine the (a) eigenvalues (b) eigenvectors for the matrix: $\begin{pmatrix} 3 & 6 \\ 1 & 4 \end{pmatrix}$

(a) The eigenvalue is determined by solving the characteristic equation $|\mathbf{A} - \lambda \mathbf{I}| = 0$

$$\text{i.e.} \quad \begin{vmatrix} 3-\lambda & 6 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

$$\text{Hence,} \quad (3-\lambda)(4-\lambda) - (6)(1) = 0$$

$$\text{i.e.} \quad 12 - 3\lambda - 4\lambda + \lambda^2 - 6 = 0$$

$$\text{and} \quad \lambda^2 - 7\lambda + 6 = 0$$

$$\text{i.e.} \quad (\lambda - 1)(\lambda - 6) = 0$$

$$\text{from which,} \quad \lambda - 1 = 0 \quad \text{i.e.} \quad \lambda = 1 \quad \text{or} \quad \lambda - 6 = 0 \quad \text{i.e.} \quad \lambda = 6$$

Hence, the eigenvalues of the matrix $\begin{pmatrix} 3 & 6 \\ 1 & 4 \end{pmatrix}$ are 1 and 6

(b) From (a) the eigenvalues of $\begin{pmatrix} 3 & 6 \\ 1 & 4 \end{pmatrix}$ are $\lambda_1 = 1$ and $\lambda_2 = 6$

Using the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda_1 = 1$

$$\text{then} \quad \begin{pmatrix} 3-1 & 6 \\ 1 & 4-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

from which,

$$2x_1 + 6x_2 = 0$$

and

$$x_1 + 3x_2 = 0$$

from which,

$$x_1 = -3x_2$$

Hence, whatever value x_2 is, the value of x_1 will be - 3 times greater. Hence the simplest

$$\text{eigenvector is: } \mathbf{x}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

Using the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda_2 = 6$

$$\text{then} \quad \begin{pmatrix} 3-6 & 6 \\ 1 & 4-6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} -3 & 6 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

from which,

$$-3x_1 + 6x_2 = 0$$

and

$$x_1 - 2x_2 = 0$$

From either of these two equations, $x_1 = 2x_2$

Hence, whatever value x_2 is, the value of x_1 will be two times greater. Hence the simplest

eigenvector is: $\mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Summarising, $\mathbf{x}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_1 = 1$

and $\mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_2 = 6$

3. Determine the (a) eigenvalues (b) eigenvectors for the matrix: $\begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$

(a) The eigenvalue is determined by solving the characteristic equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$

$$\text{i.e.} \quad \begin{vmatrix} 3-\lambda & 1 \\ -2 & 0-\lambda \end{vmatrix} = 0$$

$$\text{Hence,} \quad (3 - \lambda)(-\lambda) - (1)(-2) = 0$$

$$\text{i.e.} \quad -3\lambda + \lambda^2 + 2 = 0$$

$$\text{and} \quad \lambda^2 - 3\lambda + 2 = 0$$

$$\text{i.e.} \quad (\lambda - 1)(\lambda - 2) = 0$$

$$\text{from which,} \quad \lambda - 1 = 0 \quad \text{i.e.} \quad \lambda = 1 \quad \text{or} \quad \lambda - 2 = 0 \quad \text{i.e.} \quad \lambda = 2$$

Hence, the eigenvalues of the matrix $\begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$ are 1 and 2

(b) From (a) the eigenvalues of $\begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$ are $\lambda_1 = 1$ and $\lambda_2 = 2$

Using the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda_1 = 1$

then
$$\begin{pmatrix} 3-1 & 1 \\ -2 & 0-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

from which,

$$2x_1 + x_2 = 0$$

and

$$-2x_1 - x_2 = 0$$

from which,

$$x_2 = -2x_1$$

Hence, whatever value x_1 is, the value of x_2 will be - 2 times greater. Hence the simplest

eigenvector is: $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Using the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda_2 = 2$

then
$$\begin{pmatrix} 3-2 & 1 \\ -2 & 0-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

from which,

$$x_1 + x_2 = 0$$

and

$$-2x_1 - 2x_2 = 0$$

From either of these two equations, $x_1 = -x_2$ or $x_2 = -x_1$

Hence, whatever value x_1 is, the value of x_2 will be - 1 times greater. Hence the simplest

eigenvector is: $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Summarising, $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_1 = 1$

and $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_2 = 2$

4. Determine the (a) eigenvalues (b) eigenvectors for the matrix: $\begin{pmatrix} -1 & -1 & 1 \\ -4 & 2 & 4 \\ -1 & 1 & 5 \end{pmatrix}$

(a) The eigenvalue is determined by solving the characteristic equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$

$$\text{i.e.} \quad \begin{vmatrix} -1-\lambda & -1 & 1 \\ -4 & 2-\lambda & 4 \\ -1 & 1 & 5-\lambda \end{vmatrix} = 0$$

Hence, using the top row:

$$(-1-\lambda)[(2-\lambda)(5-\lambda) - (4)(1)] - 1[-4(5-\lambda) - (4)(-1)] + 1[(-4)(1) - (-1)(2-\lambda)] = 0$$

$$\text{i.e.} \quad (-1-\lambda)[10 - 2\lambda - 5\lambda + \lambda^2 - 4] + 1[-20 + 4\lambda + 4] + 1[-4 + 2 - \lambda] = 0$$

$$\text{i.e.} \quad (-1-\lambda)[\lambda^2 - 7\lambda + 6] - 16 + 4\lambda - 2 - \lambda = 0$$

$$\text{and} \quad -\lambda^2 + 7\lambda - 6 - \lambda^3 + 7\lambda^2 - 6\lambda - 16 + 4\lambda - 2 - \lambda = 0$$

$$\text{i.e.} \quad -\lambda^3 + 6\lambda^2 + 4\lambda - 24 = 0$$

Using the factor theorem or an electronic calculator,

$$\lambda = 2, \lambda = 6 \text{ or } \lambda = -2$$

Hence, the eigenvalues of the matrix $\begin{pmatrix} -1 & -1 & 1 \\ -4 & 2 & 4 \\ -1 & 1 & 5 \end{pmatrix}$ are 2, 6 and -2

(b) From (a), the eigenvalues of $\begin{pmatrix} -1 & -1 & 1 \\ -4 & 2 & 4 \\ -1 & 1 & 5 \end{pmatrix}$ are $\lambda_1 = 2$, $\lambda_2 = 6$ and $\lambda_3 = -2$

Using the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda_1 = 2$

$$\text{then} \quad \begin{pmatrix} -1-2 & -1 & 1 \\ -4 & 2-2 & 4 \\ -1 & 1 & 5-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} -3 & -1 & 1 \\ -4 & 0 & 4 \\ -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which,

$$-3x_1 - x_2 + x_3 = 0$$

$$-4x_1 + 4x_3 = 0$$

$$-x_1 + x_2 + 3x_3 = 0$$

From the second equation,

$$x_1 = x_3$$

Substituting in the first equation,

$$-3x_1 - x_2 + x_1 = 0 \quad \text{i.e.} \quad -2x_1 = x_2$$

Hence, when $x_1 = 1$, $x_2 = -2$ and $x_3 = 1$

Hence the simplest eigenvector corresponding to $\lambda_1 = 2$ is: $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Using the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda_2 = 6$

$$\text{then} \quad \begin{pmatrix} -1-6 & -1 & 1 \\ -4 & 2-6 & 4 \\ -1 & 1 & 5-6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} -7 & -1 & 1 \\ -4 & -4 & 4 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which,

$$-7x_1 - x_2 + x_3 = 0$$

$$4x_1 - 4x_2 + 4x_3 = 0$$

$$-x_1 + x_2 - x_3 = 0$$

From the third equation,

$$x_1 = x_2 - x_3$$

Substituting in the first equation,

$$-7(x_2 - x_3) - x_2 + x_3 = 0 \quad \text{i.e.} \quad -8x_2 + 8x_3 = 0 \quad \text{i.e.} \quad x_2 = x_3$$

Hence, $x_1 = x_2 - x_3 = 0$

Hence, if $x_2 = 1$, then $x_3 = 1$ and $x_1 = 0$

Hence the simplest eigenvector corresponding to $\lambda_2 = 6$ is: $\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Using the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda_3 = -2$

then
$$\begin{pmatrix} -1-2 & -1 & 1 \\ -4 & 2-2 & 4 \\ -1 & 1 & 5-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 1 & -1 & 1 \\ -4 & 4 & 4 \\ -1 & 1 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which,

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -4x_1 + 4x_2 + 4x_3 &= 0 \\ -x_1 + x_2 + 7x_3 &= 0 \end{aligned}$$

From the second equation,

$$x_1 = x_2 + x_3$$

Substituting in the first equation,

$$(x_2 + x_3) - x_2 + x_3 = 0 \quad \text{i.e.} \quad x_3 = 0 \quad \text{and} \quad x_1 = x_2$$

Hence, if $x_1 = 1$, then $x_2 = 1$ and $x_3 = 0$

Hence the simplest eigenvector corresponding to $\lambda_2 = -2$ is: $\mathbf{x}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

5. Determine the (a) eigenvalues (b) eigenvectors for the matrix: $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

(a) The eigenvalue is determined by solving the characteristic equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$

i.e.
$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$$

Hence, using the top row:

$$(1-\lambda)[(2-\lambda)(1-\lambda) - (-1)(-1)] - 1[-1(1-\lambda) - (-1)(0)] + 0 = 0$$

i.e.
$$(1-\lambda)[2-2\lambda-\lambda+\lambda^2-1] + 1[-1+\lambda] = 0$$

i.e.
$$(1-\lambda)[\lambda^2-3\lambda+1] - 1+\lambda = 0$$

and
$$\lambda^2-3\lambda+1-\lambda^3+3\lambda^2-\lambda-1+\lambda=0$$

i.e. $-\lambda^3 + 4\lambda^2 - 3\lambda = 0$

i.e. $\lambda^3 - 4\lambda^2 + 3\lambda = 0$

i.e. $\lambda(\lambda^2 - 4\lambda + 3) = 0$

and $\lambda(\lambda - 1)(\lambda - 3) = 0$

Hence, $\lambda = 0, \lambda = 1$ or $\lambda = 3$

Hence, the eigenvalues of the matrix $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ are 0, 1 and 3

(b) From (a), the eigenvalues of $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ are $\lambda_1 = 0, \lambda_2 = 1$ and $\lambda_3 = 3$

Using the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda_1 = 0$

then $\begin{pmatrix} 1-0 & -1 & 0 \\ -1 & 2-0 & -1 \\ 0 & -1 & 1-0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

i.e.

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which,

$$\begin{aligned} x_1 - x_2 &= 0 \\ -x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + x_3 &= 0 \end{aligned}$$

From the first equation,

$$x_1 = x_2$$

From the third equation,

$$x_2 = x_3$$

Hence, when $x_1 = 1, x_2 = 1$ and $x_3 = 1$

Hence the simplest eigenvector corresponding to $\lambda_1 = 0$ is: $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Using the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda_2 = 1$

then
$$\begin{pmatrix} 1-1 & -1 & 0 \\ -1 & 2-1 & -1 \\ 0 & -1 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which,

$$\begin{aligned} -x_2 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \\ -x_2 &= 0 \end{aligned}$$

From the second equation,

$$-x_1 = x_3 \quad \text{since } x_2 = 0$$

Hence, if $x_1 = 1$, then $x_3 = -1$ and $x_2 = 0$

Hence the simplest eigenvector corresponding to $\lambda_2 = 1$ is: $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Using the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda_2 = 3$

then
$$\begin{pmatrix} 1-3 & -1 & 0 \\ -1 & 2-3 & -1 \\ 0 & -1 & 1-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which,

$$\begin{aligned} -2x_1 - x_2 &= 0 \\ -x_1 - x_2 - x_3 &= 0 \\ -x_2 - 2x_3 &= 0 \end{aligned}$$

From the first equation,

$$-2x_1 = x_2$$

Substituting in the second equation,

$$-x_1 - (-2x_1) - x_3 = 0 \quad \text{i.e.} \quad x_1 = x_3$$

Hence, if $x_1 = 1$, then $x_2 = -2$ and $x_3 = 1$

Hence the simplest eigenvector corresponding to $\lambda_3 = 3$ is: $\mathbf{x}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

6. Determine the (a) eigenvalues (b) eigenvectors for the matrix: $\begin{pmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

(a) The eigenvalue is determined by solving the characteristic equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$

i.e.
$$\begin{vmatrix} 2-\lambda & 2 & -2 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

Hence, using the top row:

$$(2 - \lambda)[(3 - \lambda)(2 - \lambda) - (1)(2)] - 2[1(2 - \lambda) - (1)(1)] + -2[(1)(2) - 1(3 - \lambda)] = 0$$

i.e.
$$(2 - \lambda)[6 - 3\lambda - 2\lambda + \lambda^2 - 2] - 2[2 - \lambda - 1] - 2[2 - 3 + \lambda] = 0$$

i.e.
$$(2 - \lambda)[\lambda^2 - 5\lambda + 4] + 2\lambda - 2 + 2 - 2\lambda = 0$$

and
$$2\lambda^2 - 10\lambda + 8 - \lambda^3 + 5\lambda^2 - 4\lambda = 0$$

i.e.
$$-\lambda^3 + 7\lambda^2 - 14\lambda + 8 = 0$$

i.e.
$$\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$$

Hence, by calculator, $\lambda = 1, \lambda = 2$ or $\lambda = 4$

Hence, the eigenvalues of the matrix $\begin{pmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ are 1, 2 and 4

(b) From (a), the eigenvalues of $\begin{pmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ are $\lambda_1 = 1, \lambda_2 = 2$ and $\lambda_3 = 4$

Using the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda_1 = 1$

then
$$\begin{pmatrix} 2-1 & 2 & -2 \\ 1 & 3-1 & 1 \\ 1 & 2 & 2-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which,

$$x_1 + 2x_2 - 2x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

Subtracting one equation from the other gives:

$$x_3 = 0$$

and from the first equation,

$$x_1 = -2x_2$$

Hence, when $x_2 = 1$, $x_1 = -2$ and $x_3 = 0$

Hence the simplest eigenvector corresponding to $\lambda_1 = 1$ is: $\mathbf{x}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

Using the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda_2 = 2$

then
$$\begin{pmatrix} 2-2 & 2 & -2 \\ 1 & 3-2 & 1 \\ 1 & 2 & 2-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 0 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which,

$$2x_2 - 2x_3 = 0$$

$$x_1 + x_2 + x_3 = 0$$

$$x_1 + 2x_2 = 0$$

From the first equation,

$$x_2 = x_3$$

From the third equation,

$$x_1 = -2x_2 \quad \text{i.e.} \quad x_1 = -2x_3$$

Hence, if $x_2 = 1$, then $x_3 = 1$ and $x_1 = -2$

Hence the simplest eigenvector corresponding to $\lambda_2 = 2$ is: $\mathbf{x}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

Using the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda_3 = 4$

then
$$\begin{pmatrix} 2-4 & 2 & -2 \\ 1 & 3-4 & 1 \\ 1 & 2 & 2-4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} -2 & 2 & -2 \\ 1 & -1 & 1 \\ 1 & 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which,

$$-2x_1 + 2x_2 - 2x_3 = 0$$

$$x_1 - x_2 + x_3 = 0$$

$$x_1 + 2x_2 - 2x_3 = 0$$

Subtracting the third equation from the first equation gives:

$$x_1 = 0$$

From the second equation,

$$x_2 = x_3 \quad \text{since} \quad x_1 = 0$$

Hence, if $x_2 = 1$, then $x_3 = 1$ and $x_1 = 0$

Hence the simplest eigenvector corresponding to $\lambda_3 = 4$ is: $\mathbf{x}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

7. Determine the (a) eigenvalues (b) eigenvectors for the matrix: $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$

(a) The eigenvalue is determined by solving the characteristic equation $|\mathbf{A} - \lambda \mathbf{I}| = 0$

$$\text{i.e.} \quad \begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{vmatrix} = 0$$

Hence, using the top row:

$$(1-\lambda)[(2-\lambda)(3-\lambda) - (1)(2)] - 1[0 - (-1)(2)] + 2[0 - (-1)(2-\lambda)] = 0$$

$$\text{i.e.} \quad (1-\lambda)[6 - 2\lambda - 3\lambda + \lambda^2 - 2] - 2 + 2[2 - \lambda] = 0$$

$$\text{i.e.} \quad (1-\lambda)[\lambda^2 - 5\lambda + 4] - 2 + 4 - 2\lambda = 0$$

$$\text{and} \quad \lambda^2 - 5\lambda + 4 - \lambda^3 + 5\lambda^2 - 4\lambda + 2 - 2\lambda = 0$$

$$\text{i.e.} \quad -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$$\text{i.e.} \quad \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

Hence, using a calculator, $\lambda = 1, \lambda = 2$ or $\lambda = 3$

Hence, the eigenvalues of the matrix $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$ are 1, 2 and 3

(b) From (a), the eigenvalues of $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$ are $\lambda_1 = 1, \lambda_2 = 2$ and $\lambda_3 = 3$

Using the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda_1 = 1$

then
$$\begin{pmatrix} 1-1 & 1 & 2 \\ 0 & 2-1 & 2 \\ -1 & 1 & 3-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which,

$$\begin{aligned} x_2 + 2x_3 &= 0 \\ -x_1 + x_2 + 2x_3 &= 0 \end{aligned}$$

From the first equation,

$$x_2 = -2x_3$$

From the third equation,

$$-x_1 + (-2x_3) + 2x_3 = 0 \quad \text{i.e. } x_1 = 0$$

Hence, when $x_3 = -1, x_2 = 2$ and $x_1 = 0$

Hence the simplest eigenvector corresponding to $\lambda_1 = 1$ is: $\mathbf{x}_1 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

Using the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda_2 = 2$

then
$$\begin{pmatrix} 1-2 & 1 & 2 \\ 0 & 2-2 & 2 \\ -1 & 1 & 3-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which,

$$-x_1 + x_2 + 2x_3 = 0$$

$$2x_3 = 0$$

$$-x_1 + x_2 + x_3 = 0$$

From the second equation,

$$x_3 = 0$$

From the first equation,

$$x_1 = x_2$$

Hence, if $x_1 = 1$, then $x_2 = 1$ and $x_3 = 0$

Hence the simplest eigenvector corresponding to $\lambda_2 = 2$ is: $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Using the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda_3 = 3$

then
$$\begin{pmatrix} 1-3 & 1 & 2 \\ 0 & 2-3 & 2 \\ -1 & 1 & 3-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which,

$$-2x_1 + x_2 + 2x_3 = 0$$

$$-x_2 + 2x_3 = 0$$

$$-x_1 + x_2 = 0$$

From the third equation,

$$x_1 = x_2$$

From the second equation,

$$x_2 = 2x_3$$

Hence, if $x_1 = 1$, then $x_2 = 1$ and $x_3 = 0.5$ or if $x_1 = 2$, then $x_2 = 2$ and $x_3 = 1$

Hence the simplest eigenvector corresponding to $\lambda_3 = 3$ is: $\mathbf{x}_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$